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Analytical investigation of thin and moderately thick-walled tubing under peripheral and biaxial loading

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ANALYTICAL INVESTIGATION OF THIN AND MODERATELY THICK-WALLED TUBING UNDER PERIPHERAL AND BIAXIAL LOADING

BY

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A

THESIS

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INTRODUCTION

The behavior under loads of thin and moderately thick walled tubing with various combinations of loadings has been investigated by many researchers. The importance of such a problem is indicated by the number of individuals who have studied tubing.

The decision to choose this subject was made after it was realized that certain industries had tubing failures which were in many ways unexplainable. The oil industry and its casing program is an example of such an industry. Their casings were sufficiently strong in the early years of oil production to meet requirements, but with the advent of longer casing strings required in deeper holes, failures became more frequent. Of course, it was simple enough to increase the thickness of casing to a point where it was strong enough to withstand the pressures and loads associated with deeper holes, but such a method led to over weight casing and a subsequent increase in cost. The more costly this procedure became, the more desirable it became to spend money for research.

One of the reasons that casing failures became more frequent with greater drilling depths was that the casing underwent a greater tensile loading due to the weight of the casing. This loading on the casing was accompanied by an increased external fluid pressure on the outside surface of the casing from the fluid pressures in the various strata below the earth's surface. The casing is usually surrounded by cement on the outside, but in some cases the cement does not always fill the entire annular space. Such being the case, the casing is subjected to the full effect of the external fluid pressure.

The problem then, resolves itself into a study of the behavior of thin-walled and moderately thick-walled cylinders under the effects of an external fluid pressure acting alone, or the effects of an external fluid pressure acting simultaneously with an axial loading. In this thesis a theoretical or analytical approach will be used. Some application and reference will be made to oil-well casings in particular, but the solutions will not be restricted to oil-well casings alone.

It must be recognized that the tubing problem involves the problem of unstability before the proportional limit of the metal is exceeded and a similar problem after the proportional limit is exceeded. This breaks down the investigation into four main divisions. The first is a discussion of the effect of external fluid pressure acting alone on tubes that fail by becoming unstable before the proportional limit of the metal is reached. The second part is similar to the first but involves the effects produced by an axial load acting in conjunction with the external pressure. These two cases complete the study of failure due to unstability at stresses below the elastic limit or within what is called the elastic range.

The next two parts of the thesis are concerned with the effects of the types of loadings mentioned before, that is, external pressure alone or with external pressure and axial loading combined, but in a stress range above the proportional limit of the metal. This range, which is for moderately thick tubes, is known as the plastic range. The limits of this range are the proportional limit and the yield point of the tubing or casing material.

REVIEW OF LITERATURE

A survey of the field of thin-walled tubing shows that a great deal of literature has been written on the subject. The various authors have shown originality and resourcefulness in their work in studying each phase of tubing design. Heretofore however, there have been few who combined under one writing so complete an analysis such as has been undertaken in this thesis. Most of the work done from a theoretical approach has been done piecemeal, i.e., only parts of the whole problem have been considered at one writing. The work of Holmquist and Nadai have come closest to the most comprehensive work on this subject. This thesis enlarges upon their work in that it provides a rational expression for the design of short thin-walled cylinders within the elastic range under either tension or compression axially and under external pressure. It also shows more concisely that in the case of casing design for the oil fields that the effect of end loads is negligible.

Prescott must be given much of the credit for having developed the mathematical derivation of the expression for thin elastic tubes under biaxial stresses but did not apply his work, as has been done here, to oil casing.

The work on collapsing pressures acting alone on thin tubes in the elastic range has been investigated as long ago as 1850. In this thesis it is applied to casing and is used here not as something original but used to make this investigation more complete and understandable.

The practical experiments conducted elsewhere for checking the theory given in the work of this thesis are very limited and suggest a large field for future investigation.

PART I: THE CASE OF ELASTIC COLLAPSE OF TUBES BY AN EXTERNAL PRESSURE ACTING ALONE

First the derivation of a formula for the buckling of thin tubes by external pressure alone in the elastic range will be considered. Referring to Figure 1 and those notations alongside the figure, the tube in the bent position will be treated as in equilibrium in a slightly deformed shape. Considering the axes OA and XOY as axes of symmetry, the compressive force at X and Y is

$$Q = p(r_m - w_0) = p\,\overline{XO}$$

The bending moment at any cross section C is

$$\begin{aligned} M &= M_0 + Q\,\overline{XF} - p\,\overline{XZ}\cdot\frac{\overline{XZ}}{2} \\ &= M_0 + p\,\overline{XO}\cdot\overline{XF} - \frac{p}{2}\,\overline{XZ}^2 \\ &= M_0 + p\left[\overline{XO}\cdot\overline{XF} - \frac{1}{2}\,\overline{XZ}^2\right]. \end{aligned} \qquad (1)$$

From the law of cosines

$$\overline{ZO}^2 = \overline{XZ}^2 + \overline{XO}^2 - 2\overline{XZ}\cdot\overline{XO}\cdot\cos\alpha,$$

but $\cos\alpha = \dfrac{\overline{XF}}{\overline{XZ}}$ so

$$\overline{ZO}^2 = \overline{XZ}^2 + \overline{XO}^2 - 2\,\overline{XF}\;\overline{XO},$$

and

$$\frac{1}{2}\left[\overline{XO}^2 - \overline{ZO}^2\right] = \overline{XF}\cdot\overline{XO} - \frac{1}{2}\,\overline{XZ}^2.$$

Substituting in equation (1),

$$M = M_0 + \frac{p}{2}\left[\overline{XO}^2 - \overline{ZO}^2\right].$$

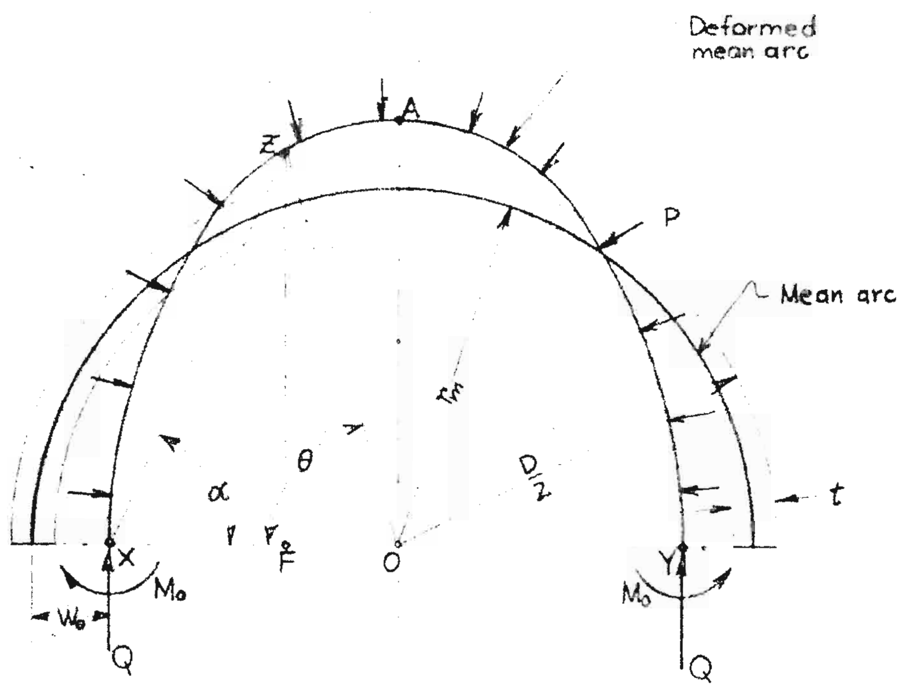


Fig. No. 1

Cross Section View of Cylinder Showing Mean Arc Before and After Deformation.

r_m - mean radius
D - outside diameter of cylinder
p - pressure per unit length
σ - compressive stress at X and Y
Q - compressive force at X and Y
w - radial displacement at any point
w_o - radial displacement at point X
M_o - bending moment acting at cross sections X and Y

With this expression for the bending moment and from figure 1,

$$\overline{XO} = r_m - w_o \quad ; \quad \overline{ZO} = r_m - w$$

then

$$M = M_o + p r_m (w - w_o). \tag{2}$$

The differential equation for the deflection curve of a long circular tube as given by Timoshenko[1] is

(1) S. Timoshenko, Theory of Elastic Stability, N.Y., McGraw Hill, 1936, p. 207

$$\frac{d^2 w}{d\theta^2} + w = -\frac{M r_m^2}{E'I}$$

where $E' = \dfrac{E}{1 - \nu^2}$

ν = Poisson's ratio

E = Young's modulus of elasticity

I = Moment of inertia

Substituting the moment expression (2) into the above equation the differential equation that follows is

$$\frac{d^2 w}{d\theta^2} + w = -\frac{r_m^2}{E'I}\left[M_o + p r_m (w - w_o)\right],$$

and

$$\frac{d^2 w}{d\theta^2} + w\left(1 + \frac{p r_m^3}{E'I}\right) = \frac{-M_o r_m^2 + p r_m^3 w_o}{E'I} \tag{3}$$

Let $\quad f^2 = 1 + \dfrac{p r_m^3}{E'I}, \qquad (4)$

then the solution of equation (3) becomes:

$$w = A\sin f\theta + B\cos f\theta + \frac{-M_o r_m^2 + p r_m^3 w_o}{E'I + p r_m^3} \tag{5}$$

The derivative of w with respect to θ is then

$$\frac{dw}{d\theta} = fA\cos f\theta - fB\sin f\theta$$

and also from symmetry

$$\left(\frac{dw}{d\theta}\right)_{\theta=0} = 0 \quad ; \quad \left(\frac{dw}{d\theta}\right)_{\theta=\frac{\pi}{2}} = 0.$$

With these two conditions it is seen then that

$$A = 0 \quad \& \quad \sin\frac{f\pi}{2} = 0.$$

Then f must be an even integer and the smallest root different from zero in this equation is when $f = 2$. This means that a critical situation occurs at some pressure, p_{cr}. This value is found when $f = 2$ is substituted in equation (4).

$$p_{cr} = \frac{3E'I}{r_m^3} \tag{6}$$

From

$$r_m = \frac{D-t}{2} \quad \text{and} \quad I = \frac{t^3}{12}$$

equation (6) becomes

$$p_{cr} = \frac{2E'}{(D/t-1)^3} = \frac{2E}{(1-\nu^2)} \cdot \frac{1}{(D/t-1)^3} \tag{7}$$

If this is a thin tube, that is, if D/t is large, a linear distribution of circumferential stress may be assumed and therefore the circumferential stress is given by

$$\sigma = \frac{pD}{2t}, \tag{7a}$$

and the stress at the critical pressure is

$$\sigma_{cr} = \frac{E D t^2}{(1-\nu^2)(D-t)^2} \quad . \tag{8}$$

The assumption that $\overline{AO}$ and $\overline{XO}$ are axes of symmetry is consistent with tests made by Sturm.[2] These equations are developed for the elastic range and are not necessarily true where the elastic limit of the material is exceeded. A typical plot of equations (7) and (8) is shown in Figure 2. These curves are called the elastic curves.

(2) Sturm, R. G., A Study of the Collapsing Pressure of Thin Walled Cylinders., Doctorate thesis, Univ. of Ill.

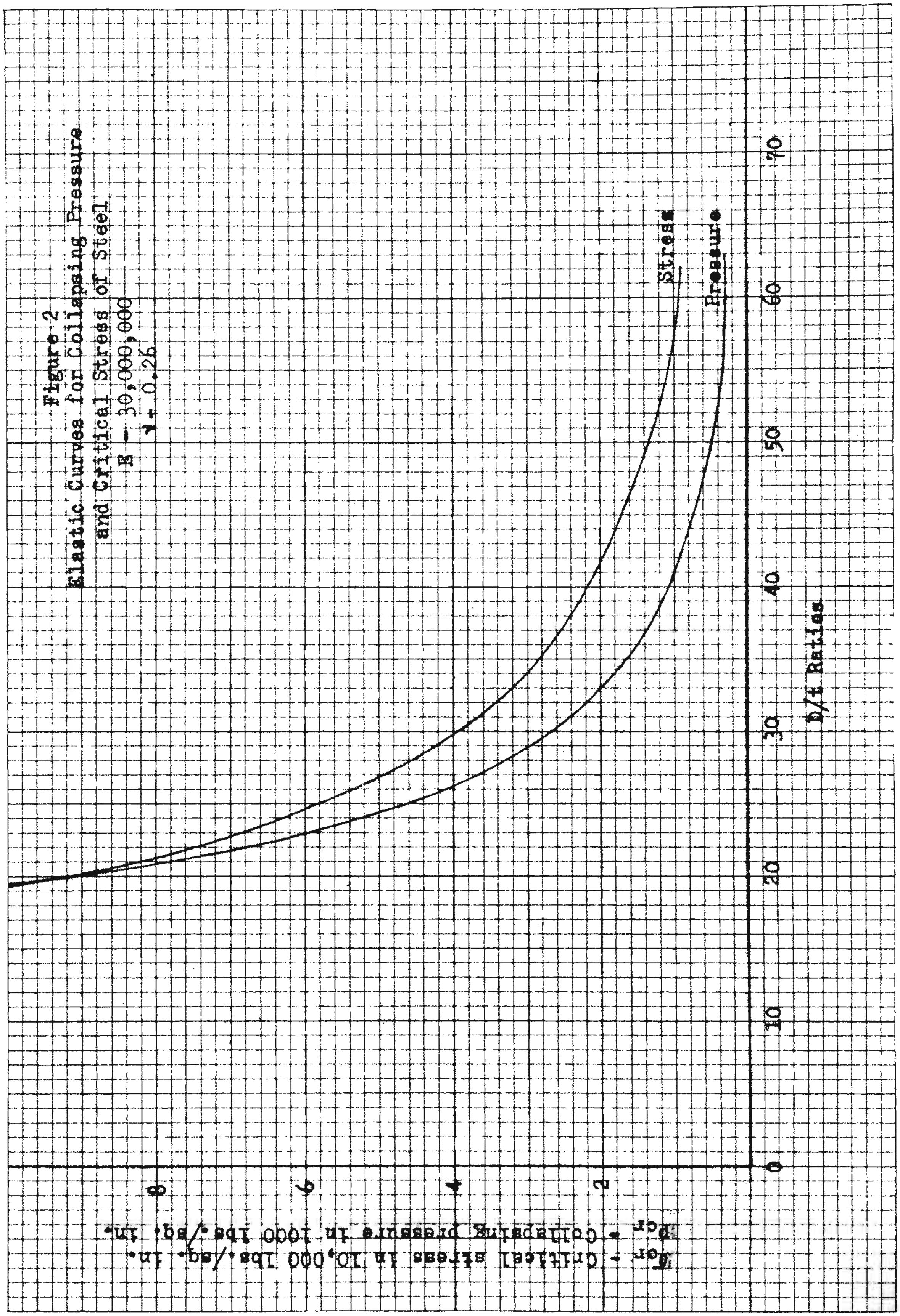


Figure 2
Elastic Curves for Collapsing Pressure
and Critical Stress of Steel
E = 30,000,000
ν = 0.26
Stress
Pressure
D/t Ratios
0
10
20
30
40
50
60
70
2
4
6
8
σcr - Critical stress in 10,000 lbs./sq. in.
pcr - Collapsing pressure in 1000 lbs./sq. in.

NOTATIONS FOR PART II

s - distance along mean circumference

x - distance along the axis of the cylinder

r_m - mean radius of cylinder

θ - angle of polar coordinate

σ_1, σ_x - stresses along longitudinal axis

σ_2, σ_s - stresses along mean circumference

u, u_o, u_1 - displacements along x-axis

w, w_o, w_1 - radial displacement

η - displacement of angle θ

p - unit external pressure

M_1, M_2 - bending moments per unit length

τ - shear stress

T - torque

F_1, F_2 - shearing forces per unit length

E - Young's modulus of elasticity

E_s - shearing modulus of elasticity

ρ - radius of curvature

c_1, c_2 - curvature

ν - Poisson's ratio

γ - shearing strain

λ - angle of twist per unit length

t - thickness of cylinder wall

ψ - angle of inclination of stresses $\sigma_1 + \sigma_x$

$d\phi$ - angle of element after strain

α - unit strain in the x - direction

β - unit strain in the direction θ

PART II: THE CASE OF ELASTIC COLLAPSE OF TUBES BY AN EXTERNAL PRESSURE ACTING SIMULTANEOUSLY WITH AN AXIAL END LOAD

For the case of a cylinder with an external pressure p, and a simultaneous axial load of either tension or compression, a general set of equations must be found. The general set of equations found in the following work will be for the case with a compressive axial load. Then with these equations and certain end conditions depending upon the application, a general expression for the collapsing pressure with the effect of an end load can be devised.

Referring to an element of cylinder in the unstrained state, an example of which is found in figure 3, a particle of the middle surface is at a position defined by x, r_m, θ. After strain the particle will be at a position $x + u_1$, $r_m + w_1$, and $\theta + \eta$ where u_1, w_1, and η are displacements of the axial distance, radial distance, and the polar angle θ respectively. Let the element have the dimensions $dx \cdot ds$. The element of middle surface is considered to be in equilibrium by the following forces, moments, and stresses acting as shown in figure 4:

$\sigma_1 + \sigma_x$ - compressive stress in the middle surface in an axial direction

$\sigma_2 + \sigma_s$ - compressive stress in the middle surface in a peripheral direction

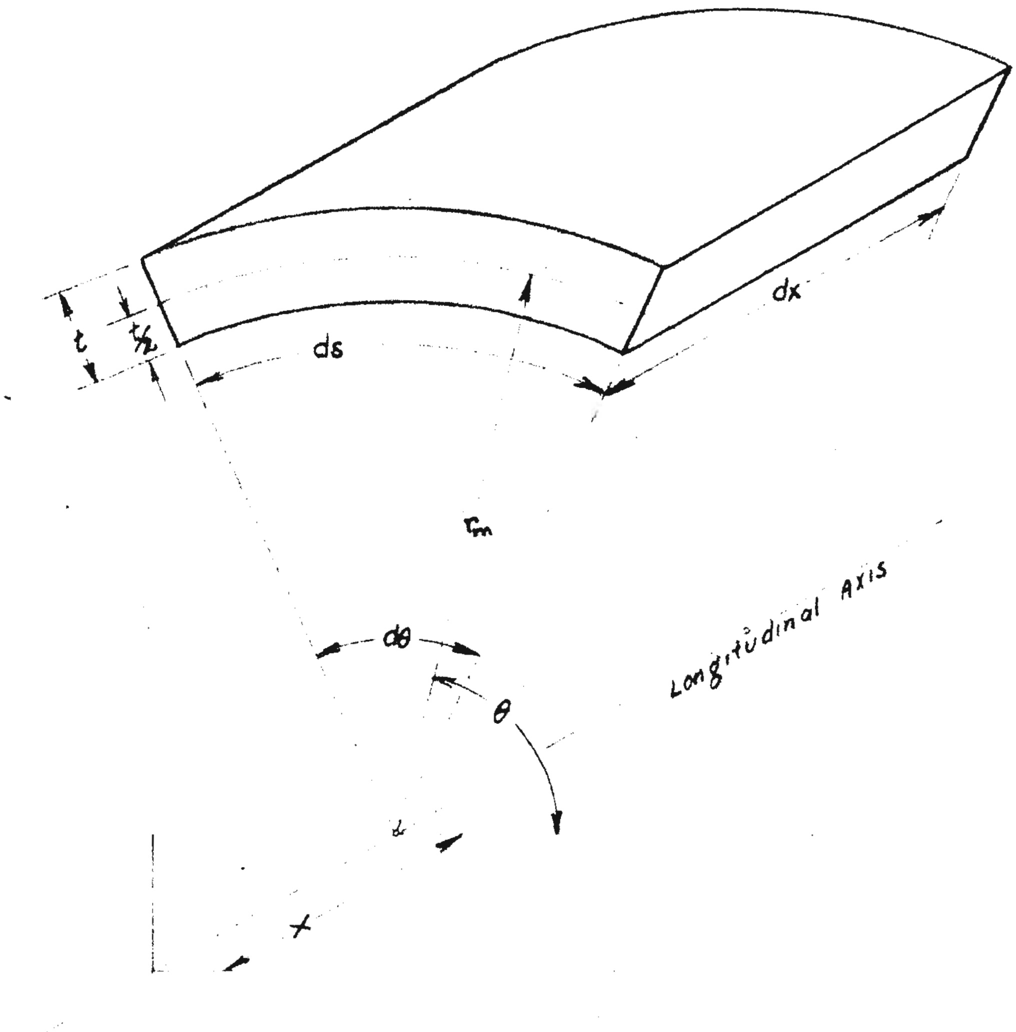
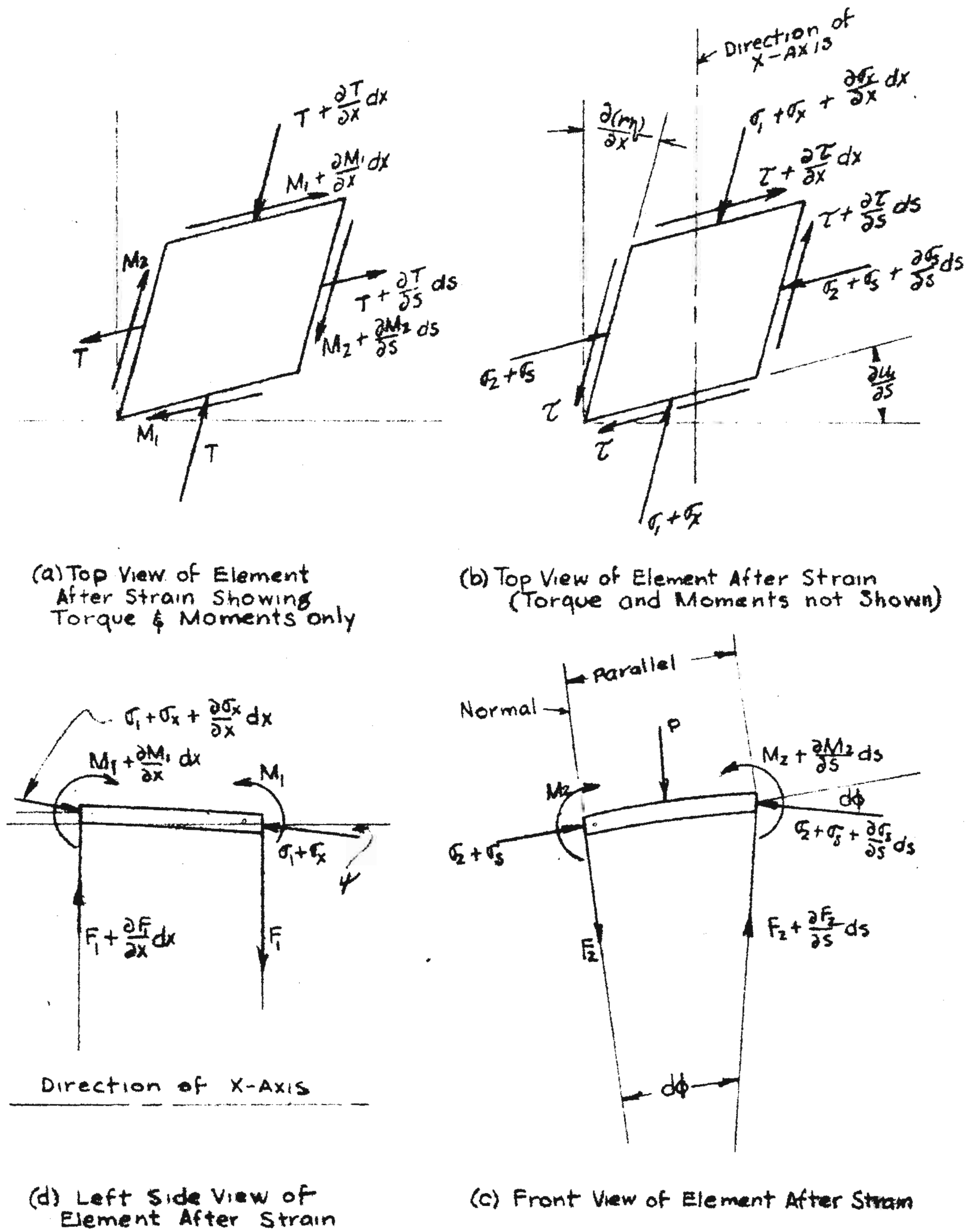


Fig. No. 3
Element of Thin Tube dx X ds. Also Longitudinal Axis of Tube



(a) Top View of Element After Strain Showing Torque & Moments only

(b) Top View of Element After Strain (Torque and Moments not Shown)

(d) Left Side View of Element After Strain

(c) Front View of Element After Strain

Fig. No. 4
Various Projections of Element in Fig. No. 3 After Strain

M_1 - bending moment per unit length acts along the same section as does $\sigma_1 + \sigma_x$

M_2 - bending moment per unit length acts along the same section as does $\sigma_2 + \sigma_s$

τ - shearing stress in the middle surface

T - torque in the middle surface

F_1 - shearing force per unit length acting on the same face as M_1

F_2 - shearing force per unit length acting on the same face as M_2

p - external pressure

ψ - angle which $\sigma_1 + \sigma_x$ makes with the x - axis

$d\phi$ - included angle of element after strain

It is to be noted that, $w_1 = w_o + w$ and $u_1 = u_o + u$, where $w_o =$ constant and $u_o =$ linear function of x. (This will be shown later). Consider also that w_o is produced solely by the stress σ_2 and that σ_1 produces u_o.

Noting figure 4b which is a view from the convex side of the element, the equation of equilibrium in an unstrained direction along the x-axis, considering the stresses acting in the middle surface always and neglecting quantities of higher order, is

$$t\,ds\frac{\partial \sigma_x}{\partial x}dx - t\,dx\frac{\partial \tau}{\partial s} - t(\sigma_2+\sigma_s)\,dx\frac{\partial^2 u}{\partial s^2}ds = 0.$$

This can be written, since σ_s is small relatively speaking, as

$$\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau}{\partial s} - \sigma_2\frac{\partial^2 u}{\partial s^2} = 0. \qquad (1)$$

Next, resolving the forces in the direction of the normal through the middle element (see figures 4c and 4d) and neglecting σ_x in comparison with σ_1, another equation of equilibrium is

$$t\, ds\, \sigma_1 \frac{\partial \psi}{\partial x} dx - dx \frac{\partial F_2}{\partial s} ds - ds \frac{\partial F_1}{\partial x} dx - t\, dx(\sigma_2 + \sigma_s)\, d\phi + p\, ds\, dx = 0$$

which in turn is

$$t \sigma_1 \frac{\partial^2 w}{\partial x^2} - \frac{\partial F_2}{\partial s} - \frac{\partial F_1}{\partial x} - t(\sigma_2 + \sigma_s)\frac{d\phi}{ds} + p = 0. \tag{2}$$

σ_s is not neglected as before because of the term $\frac{d\phi}{ds}$ which may be large.

The equation of equilibrium in the unstrained direction ds, after neglecting unimportant terms, is, from figures 4c and 4d

$$t \frac{\partial \sigma_s}{\partial s} + t r_m \sigma_1 \frac{\partial^2 \eta}{\partial x^2} - t \frac{\partial \tau}{\partial x} - F_2 \frac{\partial \phi}{\partial s} = 0. \tag{3}$$

Referring to figure 4a the summation of moments about the edge dx is

$$\frac{\partial M_\nu}{\partial s} + \frac{\partial T}{\partial x} + F_2 = 0. \tag{4}$$

The summation of moments about the edge ds is

$$\frac{\partial M_1}{\partial x} + \frac{\partial T}{\partial s} + F_1 = 0. \tag{5}$$

The expression $\frac{d\phi}{ds}$ is the curvature of the strained circumferential element which will be shown to be

$$\frac{d\phi}{ds} = \frac{1}{r_m}\left[1 - \frac{w_1}{r_m} - \frac{1}{r_m}\frac{\partial^2 w_1}{\partial \theta^2}\right]. \tag{6}$$

Writing the above equation approximately then

$$\frac{d\phi}{ds} = \frac{1}{r_m + w_0} - \frac{1}{r_m^2}\left(w + \frac{\partial^2 w}{\partial \theta^2}\right). \tag{7}$$

Neglecting quantities of higher order, the following expressions using the above equations can be written

$$(\sigma_2+\sigma_s)\frac{d\phi}{ds} = \frac{\sigma_2+\sigma_s}{r_m+w_0} - \frac{\sigma_2}{r_m^2}\left(w + \frac{\partial^2 w}{\partial\theta^2}\right), \tag{8}$$

$$F_2\frac{d\phi}{ds} = \frac{F_2}{r_m+w_0} = \frac{F_2}{r_m} \quad \text{nearly.} \tag{9}$$

Since σ_2 is the circumferential stress while the tube is a circular cylinder of (r_m+w_0), that is, while w, F_1, F_2, are all zero, and since equation (2) must remain true for this particular condition of the tube, it follows from that equation and equation (8) that

$$p = \frac{t\sigma_2}{r_m+w_0} \tag{10}$$

Saying that $r_m = (r_m+w_0)$ approximately, equation (2) becomes with the aid of (8) and (10)

$$t\sigma_1\frac{\partial^2 w}{\partial x^2} - \frac{1}{r_m}\frac{\partial F_2}{\partial\theta} - \frac{\partial F_1}{\partial x} - \frac{t\sigma_s}{r_m} + \frac{t\sigma_2}{r_m^2}\left(w+\frac{\partial^2 w}{\partial\theta^2}\right) = 0 \tag{11}$$

Also from equation (9) equation (3) becomes

$$\frac{t}{r_m}\frac{\partial\sigma_s}{\partial\theta} + t r_m\sigma_1\frac{\partial^2\eta}{\partial x^2} - t\frac{\partial\tau}{\partial x} - \frac{F_2}{r_m} = 0 \tag{12}$$

Solving for F_2 in equation (4) and F_1 in equation (5) and substituting them in equation (11) and (12) the following expressions are derived:

$$t\sigma_1 \frac{\partial^2 w}{\partial x^2} + \frac{1}{r_m^2}\frac{\partial^2 M_2}{\partial \theta^2} + \frac{2}{r_m}\frac{\partial^2 T}{\partial \theta \partial x} + \frac{\partial^2 M_1}{\partial x^2} - \frac{t\sigma_s}{r_m} + \frac{t\sigma_2}{r_m^2}\left(w + \frac{\partial^2 w}{\partial \theta^2}\right) = 0 \quad (13)$$

$$\frac{t}{r_m}\frac{\partial \sigma_s}{\partial \theta} + t r_m \sigma_1 \frac{\partial^2 \eta}{\partial x^2} - t\frac{\partial \tau}{\partial x} + \frac{1}{r_m^2}\frac{\partial M_2}{\partial \theta} + \frac{1}{r_m}\frac{\partial T}{\partial x} = 0. \quad (14)$$

The important equations of equilibrium are then equations (13) and (14) and (1) written as follows

$$\frac{\partial \sigma_x}{\partial x} - \frac{1}{r_m}\frac{\partial \tau}{\partial \theta} - \sigma_2 \frac{1}{r_m^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \quad (15)$$

With the equations of equilibrium now decided upon, expressions for the displacements will be written next. Remembering that σ_1 and σ_2 are assumed to be solely responsible for the displacements u_o and w_o respectively and that these stresses are constant, then

$$-E\frac{\partial u_o}{\partial x} = (\sigma_1 - \nu\sigma_2) = \text{constant}, \quad (16)$$

$$-E\frac{w_o}{r_m} = (\sigma_2 - \nu\sigma_1) = \text{constant}. \quad (17)$$

This shows that u_o is a linear function of x, and w_o is a constant as stated before in the opening paragraphs of Part II. The longitudinal strain of the middle surface in the x - direction is

$$\alpha = \frac{\partial u_1}{\partial x} = \frac{\partial u_o}{\partial x} + \frac{\partial u}{\partial x}; \quad (18)$$

in the direction θ it is

$$\beta = \lim_{d\theta \to 0} \frac{(r_m + w_1)\,d(\theta + \eta) - r_m\,d\theta}{r_m\,d\theta} = \frac{w_0}{r_m} + \frac{w}{r_m} + \frac{\partial \eta}{\partial \theta}. \tag{19}$$

The shearing strain is

$$\gamma = r_m \frac{\partial \eta}{\partial x} + \frac{1}{r_m}\frac{\partial u}{\partial \theta}, \tag{20}$$

since u_o is not a function of θ.

From the general formula for the curvature of a curve at polar coordinates (r,θ) and by neglecting the squares of differential coefficients of r the general formula becomes, where ρ is the radius of curvature,

$$\frac{1}{\rho} = \frac{1}{r}\left(1 - \frac{1}{r}\frac{d^2 r}{d\theta^2}\right).$$

In the particular problem at hand $r = r_m + w_1$ after strain. Therefore the curvature after strain along the circumference where M_2 acts is

$$\begin{aligned} \frac{1}{\rho} &= \frac{1}{r_m + w_1}\left[1 - \frac{1}{r_m + w_1}\frac{\partial^2 (r_m + w_1)}{\partial \theta^2}\right] \\ &= \frac{1}{r_m + w_1}\left[1 - \frac{1}{r_m + w_1}\frac{\partial^2 w_1}{\partial \theta^2}\right] \\ &= \frac{1}{r_m}\left[1 - \frac{w_1}{r_m} - \frac{1}{r_m}\frac{\partial^2 w_1}{\partial \theta^2}\right]. \end{aligned} \tag{21}$$

Equation (21) is found by approximations and has been used before in the form of equation (6). The general formulas for the curvature in cartesian coordinates are

$$\frac{1}{\rho} = \frac{d^2 y / dx^2}{\left[1 + \left(dy/dx\right)^2\right]^{3/2}}$$ and

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \qquad \text{approximately.}$$

The curvature along the tube in the direction of the x-axis where M_1 acts, after strain is

$$\frac{1}{\rho} = \frac{\partial^2 w_1}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}. \tag{22}$$

The change in curvature c_1 with M_1 positive along the x-axis is

$$c_1 = \frac{\partial^2 w}{\partial x^2} - 0 = \frac{\partial^2 w}{\partial x^2}. \tag{23}$$

from (22). Also from (21) the change in curvature along the circumferential direction with M_2 positive is

$$c_2 = \frac{1}{r_m} - \frac{1}{r_m}\left(1 - \frac{w_1}{r_m} - \frac{1}{r_m}\frac{\partial^2 w_1}{\partial \theta^2}\right)$$

$$= \frac{1}{r_m^2}\left(w_1 + \frac{\partial^2 w_1}{\partial \theta^2}\right) = \frac{1}{r_m^2}\left(w_0 + w + \frac{\partial^2 w}{\partial \theta^2}\right). \tag{24}$$

The equation for the angle of twist per unit length according to Prescott(3) is

(3) J. Prescott, Applied Elasticity. London, Longmans, Green and Co., 1924 p. 547

$$\lambda = \frac{1}{r_m}\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial \eta}{\partial x}. \tag{25}$$

The relations between stresses and strains are

$$-E\alpha = \sigma_1 + \sigma_x - \nu(\sigma_2 + \sigma_s)$$

$$-E\beta = \sigma_2 + \sigma_s - \nu(\sigma_1 + \sigma_x)$$

$$E_s \gamma = \tau$$

Combining these equations with (16) and (17) then the above equations become

$$-E\frac{\partial u}{\partial x} = \sigma_1 - \nu\sigma_2 \tag{26}$$

$$-E\left(\frac{w}{r_m} + \frac{\partial \eta}{\partial \theta}\right) = \sigma_2 - \nu\sigma_1 \tag{27}$$

$$E_s\left(r_m \frac{\partial \eta}{\partial x} + \frac{1}{r_m}\frac{\partial u}{\partial \theta}\right) = \tau \tag{28}$$

It can be shown from the theory of thin plates that

$$M_1 = E'I(C_1 + \nu C_2) \tag{29}$$

$$M_2 = E'I(C_2 + \nu C_1) \tag{30}$$

also that

$$T = 2E_s I\lambda = (1-\nu)E'I\lambda \tag{31}$$

Equations 26, 27, 28, 29, 30, and 31 are expressions for the displacements.

Now with the equation of displacements and equilibrium found, it remains to solve these equations. According to the accepted mathematical theory,[4] all the displacements

(4) Prescott, op. cit., p. 552

are functions of the coordinate x and the polar angle θ, when the tube begins to buckle. The displacements can then be expressed by

$$\left.\begin{aligned} u &= A \cos n\theta \cos \frac{kx}{r_m} \\ r_m \eta &= B \sin n\theta \sin \frac{kx}{r_m} \\ w &= C \cos n\theta \sin \frac{kx}{r_m} \end{aligned}\right\} \qquad (32)$$

The constant k depends on the end conditions of the tube. It will be noticed that w, the radial displacement, will have the same value at $\theta = 0$ and $\theta = 2\pi$. Therefore it is apparent that n must be an even integer. Substitution is now made into equations 26 through 31 which yield the following equations.

$$\sigma_x = \frac{E'}{r_m}(kA - \nu C - \nu nB) \cos n\theta \sin \frac{kx}{r_m}$$

$$\sigma_s = \frac{E'}{r_m}(\nu kA - C - nB) \cos n\theta \sin \frac{kx}{r_m}$$

$$\tau = \frac{1}{2}(1-\nu)\frac{E'}{r_m}(kB - nA) \sin n\theta \cos \frac{kx}{r_m}$$

$$M_1 = \frac{E'I}{r_m^2}\left\{(1-n^2) - \nu k^2\right\} C \cos n\theta \sin \frac{kx}{r_m} + \frac{E'I}{r_m^2}\nu w_0$$

$$M_2 = \frac{E'I}{r_m^2}\left\{(1-n^2) - \nu k^2\right\} C \cos n\theta \sin \frac{kx}{r_m} + \frac{E'I}{r_m^2} w_0$$

$$T = (1-\nu)\frac{E'I}{r_m^2}\left\{-nkC - kB\right\} \sin n\theta \cos \frac{kx}{r_m}$$

With these values and the equations of equilibrium 13, 14, and 15 the following three equations can be written:

$$\left\{n^2\frac{\sigma_2}{E'}+k^2+\frac{1}{2}(1-\nu)n^2\right\}A-\frac{1}{2}(1+\nu)nkB-\nu kC=0 \qquad (33)$$

$$-\nu kA+\left\{2(1-\nu)nk^2\frac{t^2}{12r_m^2}+n\right\}B-\left\{k^2\frac{\sigma_1}{E'}+(n^2-1)\frac{\sigma_2}{E'}-1\right\}C$$
$$+\frac{t^2}{12r_m^2}\left\{(n^2+k^2)^2-n^2-\nu k^2\right\}C=0 \qquad (34)$$

$$-\frac{1}{2}(1+\nu)nkA+\left\{-k^2\frac{\sigma_2}{E'}+n^2+\frac{1}{2}(1-\nu)\left(1+\frac{2t^2}{12r_m^2}\right)k^2\right\}B$$
$$+nC+\frac{t^2}{12r_m^2}\left\{n(n^2-1)+nk^2\right\}C=0 \qquad (35)$$

From the last three simultaneous equations, A, B, and C can be eliminated. This is done by determinants and gives the following relationship. This relationship is the result after neglecting squares and products of the extremely small quantities $\frac{\sigma_2}{E'}$, $\frac{\sigma_1}{E'}$, and $\frac{t^2}{12r_m^2}$

$$-\frac{1}{2}(1-\nu)k^2\frac{\sigma_1}{E'}\left\{(n^2+k^2)^2+n^2+2k^2+2\nu k^2\right\}$$
$$-\frac{1}{2}(1-\nu)\frac{\sigma_2}{E'}\left\{(n^2-1)(n^2+k^2)^2-n^2k^2\right\}$$
$$+\frac{1}{2}(1-\nu)\frac{t^2}{12r_m^2}\left\{(n^2+k^2)^4+n^4+3n^2k^2+2(1-\nu)k^4\right\}$$
$$+\frac{1}{2}(1-\nu)\frac{t^2}{12r_m^2}\left\{-2n^6-7n^4k^2-(7+\nu-2\nu^2)n^2k^4-\nu k^6\right\}$$
$$+\frac{1}{2}(1-\nu)(1-\nu^2)k^4=0 \qquad (36)$$

Treating k as a small fraction, as it would be in most practical cases, and arbitrarily selecting the important terms of equation 36, the resulting equation may be considered to be correct

$$\frac{\sigma_z}{E'} = \frac{(1-\nu^2)k^4}{n^4(n^2-1)} + (n^2-1)\frac{t^2}{12r_m^2} - \frac{k^2(n^2+1)}{n^2(n^2-1)}\frac{\sigma_1}{E'} . \qquad (37)$$

Equation 37 can be modified to include the value of the external pressure by assuming a linear distribution of stress σ_z on a longitudinal section which is given by

$$\sigma_z = \frac{p(D-t)}{2t}$$

which was shown before by equation 10. Then equation 37 becomes, solving for pressure

$$p = E'\frac{2t}{(D-t)}\left[\frac{(1-\nu^2)k^4}{n^4(n^2-1)} + (n^2-1)\frac{t^2}{12r_m^2} - \frac{k^2(n^2+1)\sigma_1}{n^2(n^2-1)E'}\right] (38)$$

It is noticed that the existence of an end thrust σ_1 causes the tube to collapse at a smaller value of pressure than if σ_1 were zero. Likewise an end load representing a tension (σ_1 substituted as a negative quantity) will increase the pressure for collapse.

The preceding equation and the results therefrom can only be realized if k is a reasonably large value, and if the D/t ratio is large. The value of k as stated before is a quantity which depends on the end conditions of the tube. If it is possible at the ends of the tube in question, to restrain the ends so that no buckling occurs at these points, then $w = 0$ at these points. With $w = 0$, k becomes from equation 32

$$k = \frac{\pi r_m}{L}$$

where L is the length of the tube. In the specific case of oil casing, the end restraints would be imposed by the couplings at the joints.

The value n is an even integer as shown before and is to be found by substituting even numbers starting with 2 in equation 38. The value of n is the value of the even number that yields the smallest pressure.

For oil casing where most of the lengths encountered are in the magnitude of 30 ft. or more, k would be small enough so that equation 38 can be reduced to

$$p = E' \frac{2t}{(D-t)} \left[n^2 - 1 \right] \frac{t^2}{12 r_m^2}$$

$$r_m = \frac{D-t}{2}$$

n = 2   for long tubes

or

$$p = \frac{2E'}{(D/t - 1)} = \frac{2E}{(1-\nu^2)} \times \frac{1}{(D/t - 1)^3}$$

This equation is the same as was derived before for a thin tube acted upon by an external pressure only. This then leads to the important conclusion that the end loads have little effect upon the collapsing pressure for long tubes in the elastic range.

PART III: THE CASE OF PLASTIC COLLAPSE OF TUBES BY AN EXTERNAL PRESSURE ACTING ALONE

Since many of the commercial tubes do not fall in the range of elastic collapse, the case of plastic collapse by an external pressure acting alone will now be considered. The tubes within this range are those which D/t ratios small enough to place them in a range of stress beyond the proportional limit. This is indicated in figure 2.

It has been shown that for the case of buckling of columns, that the substitution of a reduced modulus of elasticity in the buckling formula will predict accurately the buckling stress in the plastic range. On this basis and since the instability formulas for both columns and tubes run closely parallel, it is seen that the similarity can be extended to substituting a reduced modulus in the collapsing formulas. With this in mind then, equations 7 and 8 of Part I can then be written as follows

$$P_{cr} = \frac{2E_r}{(1-\nu^2)} \times \frac{1}{(D/t - 1)^3} \qquad (1)$$

$$\sigma_{cr} = \frac{E_r D t^2}{(1-\nu^2)(D-t)^3} \qquad (2)$$

where

$$E_r = \frac{4 E E'}{\left[\sqrt{E} + \sqrt{E'}\right]^2} \qquad (3)$$

E_r = reduced modulus

E = Young's modulus

E' = slope of stress-strain curve at a point on the curve

All other symbols the same as before

From the preceding it is apparent that the critical pressure and stress will depend upon the nature of the stress -strain curve beyond the proportional limit. The exact variation of this curve in the plastic range for oil field casing would have to be known if E_r were to be calculated. For purposes of clarity two examples in the use of equations 1 and 2 will be shown. For these examples it will be proposed that the steels behave in accordance with stress-strain curves as shown in figure 5. Steel no. 1 has a parabolic stress distribution beyond the proportional limit and steel no. 2 has a straight horizontal line relationship beyond the same point.

Both steels are to have a proportional limit stress of 40×10^3 psi and steel no. 1 is to have a yield strength of 60×10^3 psi as defined by a 0.2% permanent set. From fig. 5 and the stress-strain curve for steel no. 1 the portion of the curve from the proportional limit to the defined yield strength can be represented by the following equation.

$$\epsilon = (\epsilon' - \epsilon_p)\left(\frac{\sigma - \sigma_p}{\sigma_y - \sigma_p}\right)^2 + \epsilon_p \qquad (4)$$

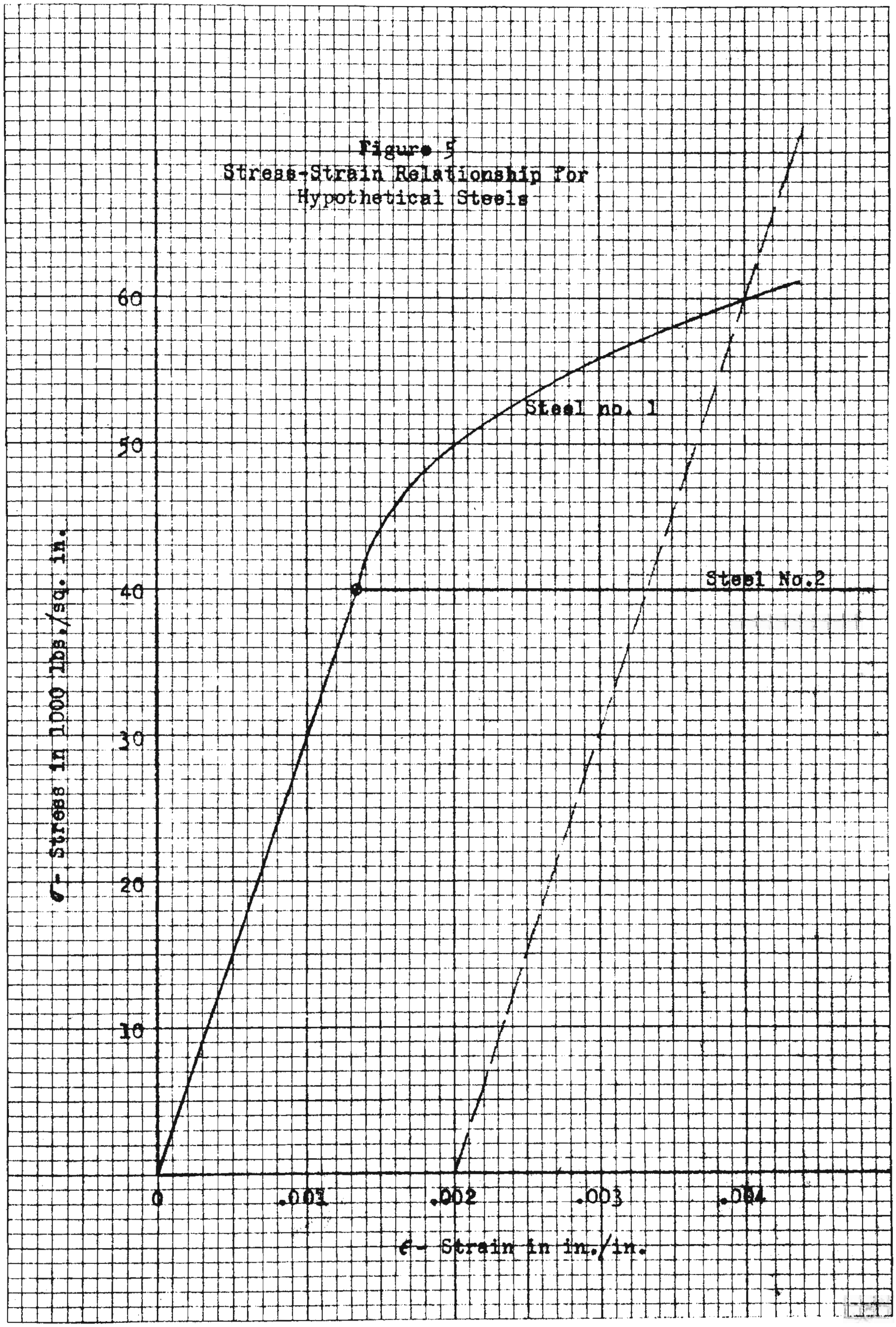


Figure 5
Stress-Strain Relationship for
Hypothetical Steels
60
50
40
30
20
10
Steel no. 1
Steel No.2
σ - Stress in 1000 lbs./sq. in.
0
.001
.002
.003
.004
ε - Strain in in./in.

The explanation of notation is

ϵ = variable strain

ϵ' = strain at the defined yield stress

ϵ_p = strain at the proportional limit

σ = variable stress

σ_p = stress at the proportional limit

σ_y = stress at the defined yield strength

The value of E' can now be found from equation 4 by finding $\frac{d\sigma}{d\epsilon}$. (Where the equation of a curve is not known, E' is found by measuring the slope of the stress-strain curve). Then the reduce modulus is solved for and substituted into equation 2. The stress curve for steel no. 1 in figure 6 is plotted from various values. Assuming, again, a linear stress distribution of the hoop stress and using formula 7a Part I, a plot of collapsing pressures can be made as shown in figure 7 for steel no. 1.

The same type of procedure is followed for steel no. 2.

Plotting the collapse pressures in the plastic range from equation 7a Part I may not always be absolutely correct since some steels do not yield homogenously. For absolutely correct curves experimental data is necessary for the particular case. Holmquist and Nadai have made some tests which show the validity of these statements. (5)

(5) J.L.Holmquist and A. Nadai, A Theoretical and Experimental Approach to the Problem of Collapse of Casing. A.P.I. Drilling and Production Practice, 1939, p. 403

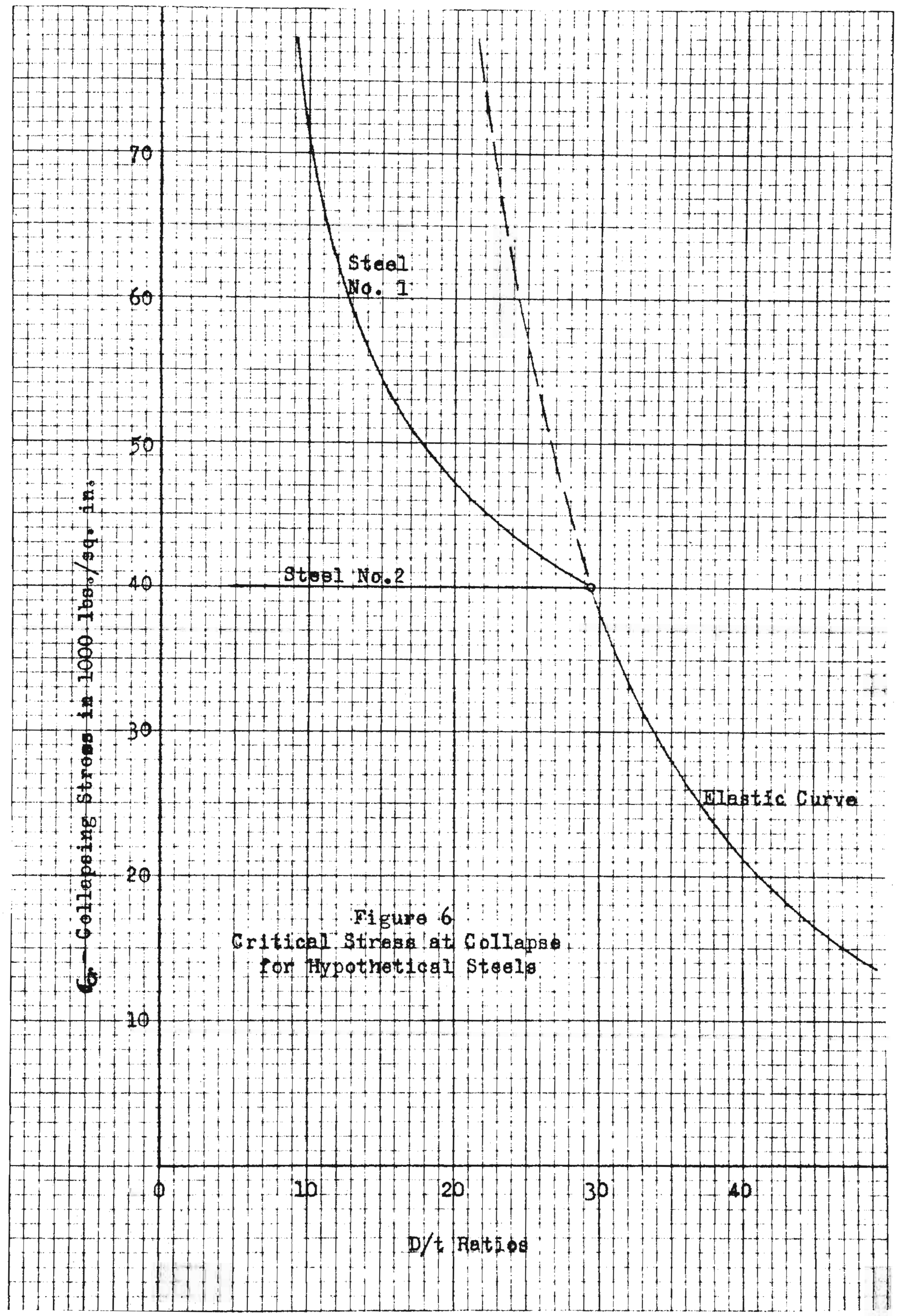


Figure 6
Critical Stress at Collapse
for Hypothetical Steels

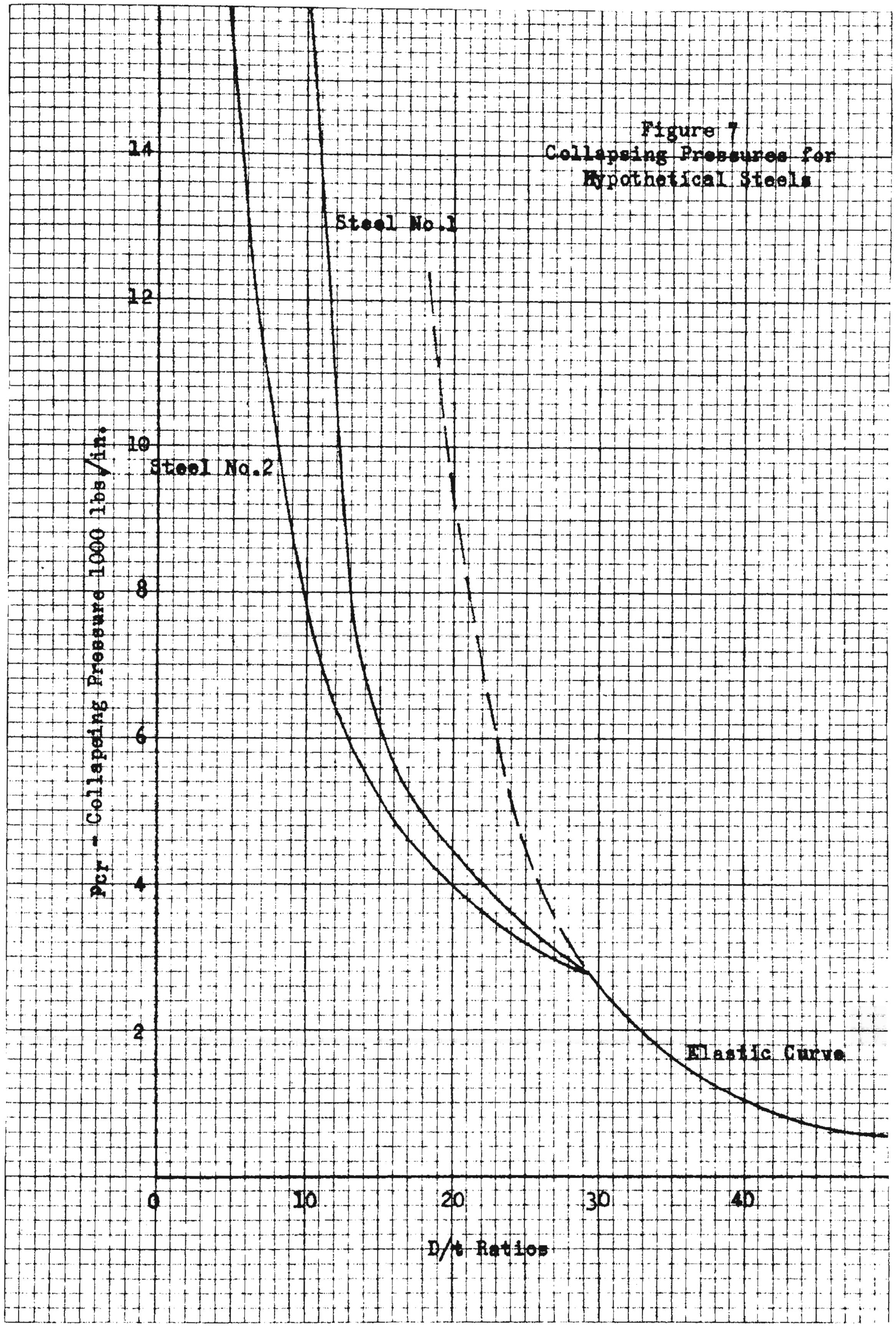


Figure 7
Collapsing Pressures for
Hypothetical Steels
Steel No.1
Steel No.2
Elastic Curve
Pcr - Collapsing Pressure 1000 lbs/in.
D/t Ratios
14
12
10
8
6
4
2
0
10
20
30
40

For Poisson's ratio in equation 2, an average value was used. For elastic strains in steel Poisson's ratio is 0.26, where as for purely plastic strains it is 0.5. The average value is 0.38. The use of an average value is tantamount to assuming a straight line variation of Poisson's ratio in the plastic range. This is an assumption which will do for the assumed problem here, but may lead to serious error otherwise. The variation of Poisson's ratio in the plastic range in buckling problems should be determined by experiment.

PART IV: YIELDING OF TUBES UNDER BIAXIAL LOADING

The effect of an axial load upon the collapse pressure and critical stress at the collapsing pressure in the plastic range for moderately thick-walled cylinders is of prime importance and will be considered under this section. This is a problem in which the consideration of some theory of failure must be taken into account. Here the yield point of the material may be lowered or raised by an axial load. This would certainly change the value of the collapsing pressure and stress since it was found before that the collapsing pressure and stress by an external pressure acting alone depended upon the yield strength in the plastic range.

To find the effect on the hoop or circumferential stress of an axial stress, consider the middle surface of the thin tube in figure 3. As before, let the axial stress be σ_1 and the hoop stress be σ_2; only this time consider them both tensile stresses. Consider the element in a state before unstability; and neglect the small effect of the radial stress.

The strain in the circumferential direction will then be

$$\epsilon_s = \frac{\sigma_2}{E} - \frac{\nu\sigma_1}{E}, \tag{1}$$

and the strain in the axial direction will be

$$\epsilon_x = \frac{\sigma_1}{E} - \frac{\nu\sigma_2}{E} \tag{2}$$

E = Young's modulus and ν = Poisson's ratio.

The total strain energy per unit volume is

$$U = \frac{\sigma_1 \epsilon_x}{2} + \frac{\sigma_2 \epsilon_s}{2} . \tag{3}$$

Substituting (1) and (2) into (3) then

$$U = \frac{\sigma_2}{2}\left[\frac{\sigma_2}{E} - \frac{\nu\sigma_1}{E}\right] + \frac{\sigma_1}{2}\left[\frac{\sigma_1}{E} - \frac{\nu\sigma_2}{E}\right]$$

$$U = \frac{1}{2E}\left(\sigma_2^2 + \sigma_1^2 - \nu\sigma_1\sigma_2\right). \tag{4}$$

The strain energy per unit volume at the yield point of the material is

$$U_y = \frac{\sigma_y^2}{2E}, \qquad \text{where } \sigma_y = \text{yield stress in pure tension.} \tag{5}$$

The strain energy theory says that the material starts to flow when $U_y = U$.

$$\frac{\sigma_y^2}{2E} = \frac{1}{2E}\left(\sigma_2^2 + \sigma_1^2 - \nu\sigma_1\sigma_2\right)$$

$$\sigma_y^2 = \sigma_2^2 + \sigma_1^2 - \nu\sigma_1\sigma_2 \tag{6}$$

The Hencky-von Mises theory says the material begins to flow when

$$\sigma_y^2 = \sigma_2^2 + \sigma_1^2 - \sigma_1\sigma_2 . \tag{7}$$

From test results this seems to be the most acceptable theory. Solving for σ_2 then

$$\sigma_2 = \frac{\sigma_1}{2} \pm \sqrt{\sigma_y^2 - \frac{3}{4}\sigma_1^2} \tag{8}$$

If the pressure on a tube is external, the hoop stress σ_2 is compressive and equation 8 becomes

$$\sigma_2 = \frac{\sigma_1}{2} - \sqrt{\sigma_y^2 - \frac{3}{4}\sigma_1^2} \tag{9}$$

σ_2 now represents the yield stress in a circumferential direction under combined external pressure and axial tension, which is less than it was when an external pressure was acting alone. The equation for the case of internal pressure and an end stress can be determined from equation 8 also. Similarily, other equations can be found from 8 if the algebraic signs are adjusted.

A plot of equation 8 is shown in figure 8 and a plot of equation 9 is shown in figure 9. Note that the curves are plotted with the ratios $\frac{\sigma_1}{\sigma_y}$ as abscissae and the ratios $\frac{\sigma_2}{\sigma_y}$ as ordinates. Since the plotted values are ratios, there are many materials which are represented by these curves.

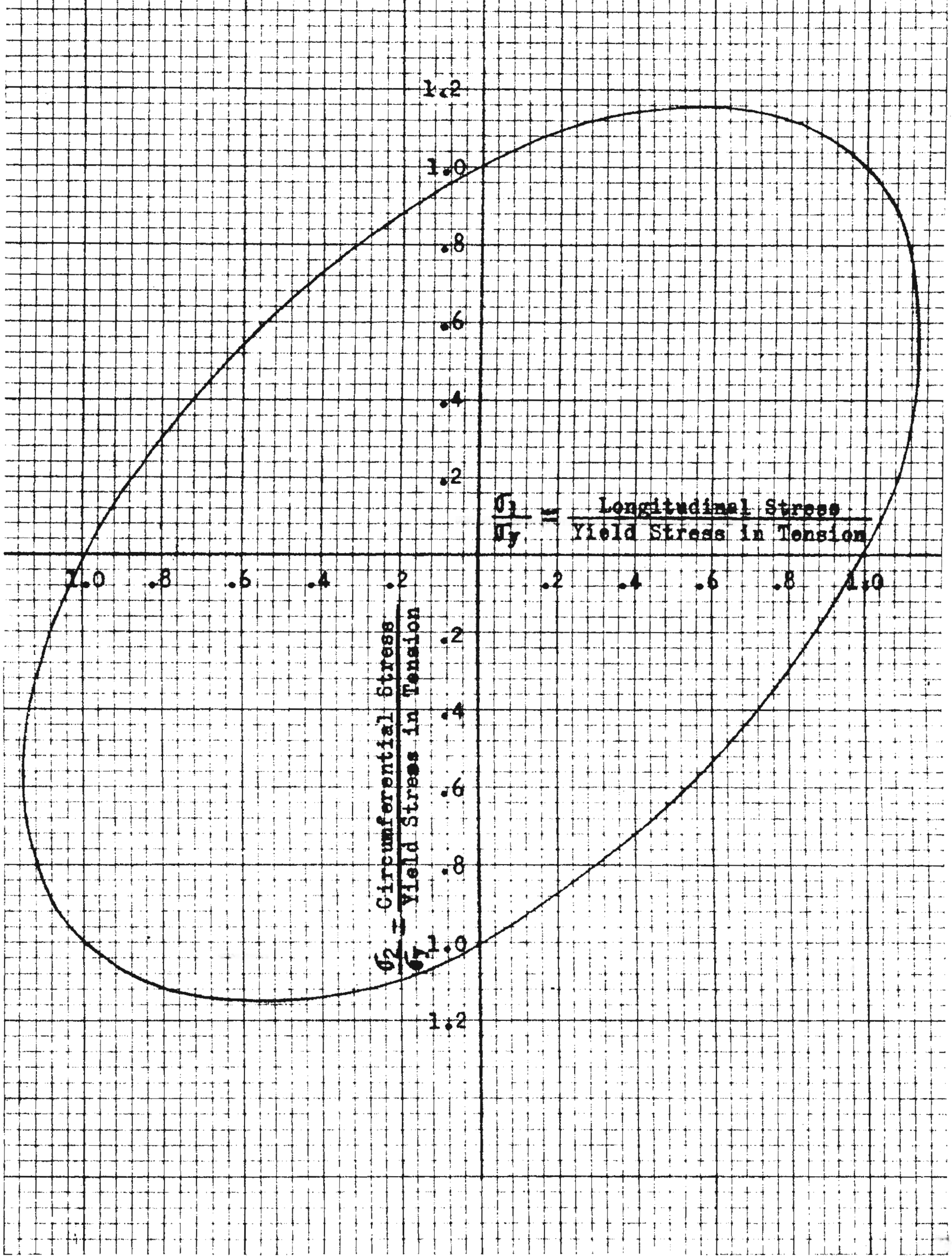
It has been shown by Edwards and Miller by experiment that some tubes behave as indicated by formulas 8 and 9.(6)

(6) S.H.Edwards and C.P.Miller, Discussion on the Effect of Combined Longitudinal Loading and External Pressure on the Strength of Oil-Well Casing, API Drilling and Production Practice, pp. 483-502, 1939.

The aforementioned investigators also showed by experiment that the substitution of the ratio $\frac{p}{p_{cr}}$ for the ordinate $\frac{\sigma_2}{\sigma_y}$ is a valid assumption, where p is the reduced collapsing pressure due to the effect of an axial stress and p_{cr} is the collapsing pressure as defined by equation 1 Part III. The ratio $\frac{p}{p_{cr}}$ multiplied by 100 gives the percent of the collapse pressure p_{cr} to be used when an axial load is applied.

This percentage is also plotted in figure 9.

Figure 8
Effect of Biaxial Loading in Tubing
on the Yield Stress
(Hencky-von Mises Theory)



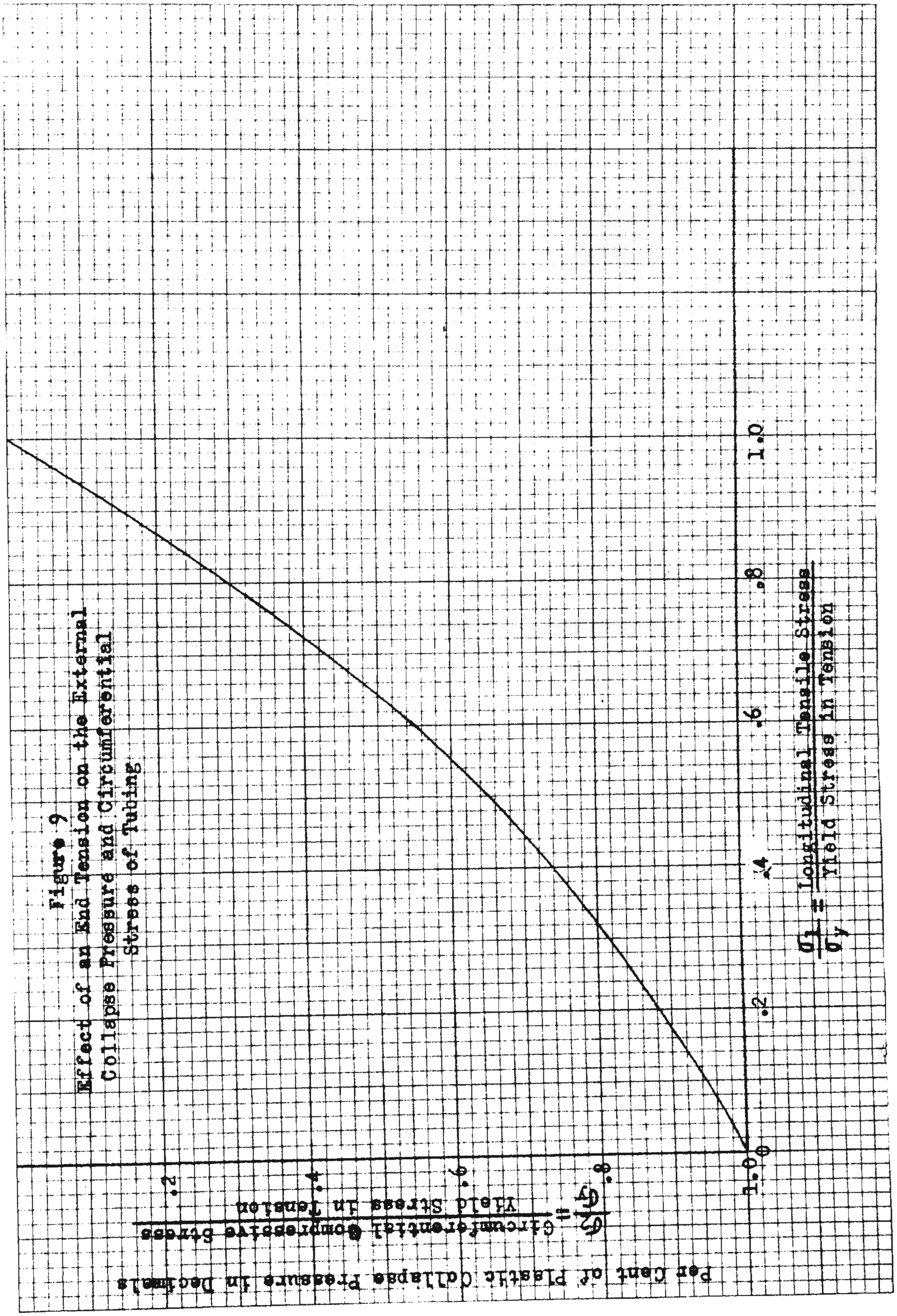


Figure 9
Effect of an End Tension on the External Collapse Pressure and Circumferential Stress of Tubing

CONCLUSIONS

An important conclusion is that tubing collapse is similar to the buckling encountered in long columns and much of the same reasoning can be applied to both problems.

It can also be concluded that in the elastic case, the effect of a compressive or tensile stress acting at the end of the tube has little effect on long tubes and should be considered in the case of short tubes. For short tubes the effect of axial end tension is to raise the collapsing external pressure and the effect of axial end compression is to lower the collapsing pressure.

In the plastic region it is important to notice that the yield strength of the tube material is lowered or raised by an end load acting simultaneously with a fluid pressure depending upon whether the fluid pressure is external or internal and whether or not it acts with an end load of tension or compression. A specific example is the case of an external pressure and an end load in tension. Here the yield strength is lowered along the circumferential direction (and thus the collapsing pressure) by the end tensile stress. This is the case of many oil well casing problems.

To prove effectively the theoretical work done here, tests should be made on the actual casing as to the actual variation of the reduced modulus of elasticity and Poisson's ratio in the plastic range. This would probably enable the petroleum industry to raise their safety factors on casing.

The results of this work show that the collapsing pressure of the moderately thick-walled casing is least affected by an end tension in the range of higher D/t ratios.

SUMMARY

The following are the important points brought out in this thesis:

1. An expression for the collapsing pressure and critical stress for the elastic case of thin tubing was derived.
2. An expression for the collapsing pressure and critical stress with an axial end compression or end tension of thin tubing was derived.
3. Recognition of the two major cases of behavior of tubing, namely, the elastic and plastic case was made.
4. An expression for the collapsing pressure and critical stress for the plastic case of moderately thick-walled tubing was found.
5. An expression for use in determining the change of the yield point in a tube under fluid pressure in the plastic range by an axial end load was found.
6. A curve to illustrate point no. 5 was plotted.
7. A curve to show the decrease in collapsing pressure and critical stresses as a result of an axial end tensile stress was plotted.

BIBLIOGRAPHY

1. Books:

    a. Prescott, J. Applied elasticity. London, Longmans, Green and Co., 1924. pp 530-564

    b. Timoshenko, S. Theory of elasticity. N.Y., McGraw-Hill, 1934. pp. 416

    c. Timoshenko, S. Theory of elastic stability. N.Y., McGraw-Hill, 1936. pp. 204-221

2. Periodicals:

    a. Clinedinst, W. O. Collapse safety factors for tapered casing strings. Oil Weekly. Vol. 118, p. 50 (June 25, 1945)

    b. Clinedinst, W. O. Drill pipe yield strength lowered by severe drilling service. Oil & Gas Journal. Vol. 45, pp 76-77 (Nov. 2, 1946)

    c. Cooley, H. M. Factors involved in design of long casing strings. Oil & Gas Journal. Vol. 42, pp. 25-28 (Jan. 20, 1944)

    d. Jasper, T. M. Correct setting depth for well casing depends on compression yield point. Oil & Gas Journal. Vol. 34, p. 81 (Nov. 14, 1935)

    e. Kemler, E. N. The design of casing strings. Oil Weekly. Vol. 103, p. 21 (Dec. 1, 1941). Vol. 104, p. 36 (Dec. 8, 1944). Vol. 104, p. 33 (Dec. 15, 1941). Vol. 104, p. 36 (Dec. 22, 1944) Vol. 104, p. 34 (Dec 29, 1941).

    f. O'Donnell, L. & Crake, W. S. Mechanical causes of casing failure and practices for their control. Oil & Gas Journal. Vol. 42, p. 46 (Dec. 16, 1943)

    g. Porter, L. E. Factors to be considered in determining proper casing size. Oil Weekly. Vol. 102, pp. 81-82 (July 14, 1941)

    h. Wais, J. Recent developments in casing standards and design. Oil Weekly. Vol. 126, pp. 37-38 (June 9, 1947)

BIBLIOGRAPHY
(cont.)

3. Publications of Learned Societies:

a. A.P.I. Information on collapsing pressures and setting depths for casing. Division of Production Bulletin No. 5-C-2. 2nd edition, Mar. (1940)

b. Blaine, B. W., Dunlop, C. A., and Kemler, E. N. Setting depths for casing. A.P.I. Drilling & Prod. Practice. pp. 125-182 (1940)

c. Clinedinst, W. O. A rational expression for the critical collapsing pressure of pipe under external pressure. A.P.I. Drilling & Prod. Practice. pp. 383-391 (1939)

d. Edwards, S. H. and Miller, C. P. Discussion on the effect of combined longitudinal loading and external pressure on the strength of oil-well casing. A.P.I. Drilling & Prod. Practice. pp. 483-502 (1939)

e. Holmquist, J. L. and Nadai, A. A theoretical and experimental approach to the problem of collapse of deep-well casing. A.P.I. Drilling and Prod. Practice. pp. 392-420 (1939)

f. Main, W. C. Combining bending and hoop stresses to determine collapsing pressure of oil-country tubular goods. A.P.I. Drilling and Prod. Practice pp. 421-431 (1939)

4. Unpublished Material:

a. Bobo, R. Developments in casing design. Paper written for production department of Phillips Petroleum Co. (1948)

b. Sturm, R. G. A study of the collapsing pressure of thin-walled cylinders. Doctorate thesis, Univ. of Ill., Urbana, Ill. (1936)

VITA

The author was born in Chicago on November 2, 1922. After his primary and secondary education, he entered Morton Jr. College of Cicero, Illinois in September 1940. Graduating in June 1942 he was later enrolled as a student in the Mechanical Engineering Department of the Illinois Institute of Technology. Upon his graduation from I.I.T. in February 1944, he went to work as a tool designer for Thompson Products Inc. of Cleveland, Ohio. Working in this plant until July 1944, he left to enter the service of the United States Naval Reserve, as a member of the Civil Engineer Corps. Approximately two years later after his separation from active duty with the navy in August 1946 he went to work as an engineer for the Master Manufacturing Co. of Chicago.

The author was then appointed instructor in mechanical engineering in February 1947 at the Missouri School of Mines and Metallurgy where he completed enough work for a Master's degree in Mechanical Engineering.